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Dynamic relationship between Stock market and Bond market:

A GAS-MIDAS copula approach

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Motivation

Optimal portfolio allocation depends on the correlation of Stocks and Bonds.

Certain changes in the dependence might anticipate a worsening of macroeconomic conditions.

Moreover, this dependence fluctuates overtime.

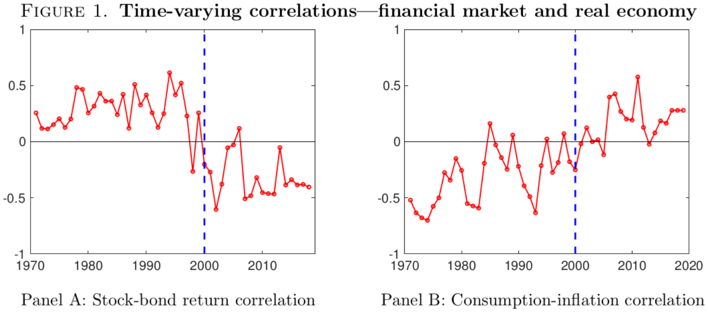


Figure: Stock return and bond return correlation Li et al. (2020)

Contributions and findings

Contributions

A Generalized Autoregressive Score (GAS) Mixed Data Sampling (MIDAS) copula model is proposed for the dynamic dependence of Stock returns and Bond returns.

Findings

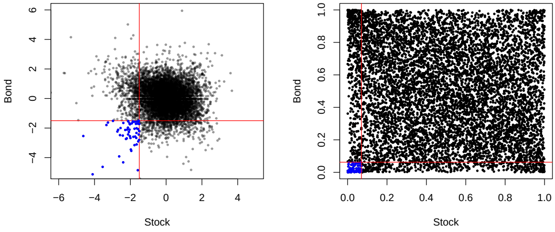
Besides the realized correlation, other economic variables can explain for the changes in the long term dependence of Stock returns and Bond returns such as Inflation and Interest rate, the state of economy, illiquidity, etc...

Moreover, the Survey of Professional Forecasters (SPF) can provide a forward looking for the changes in the dependence.

1. Introduction to copulas
2. Econometric models
3. Empirical illustration
4. Conclusion

Introduction to Copulas

Copula is a n-dimensional joint cumulative distribution function (cdf) in the unit domains.



Let $F(x_1, ..., x_d|\theta)$ be a n-dimensional joint cdf with marginals $F_1, ..., F_d$ for all x_i in $[-\infty, \infty]$, and $u_i = F_i(x_i|\theta_i)$ for all $i = 1, ..., d$, (see Sklar (1959))

$$F(x_1, ..., x_d|\theta) = C(u_1, ..., u_d|\theta)$$

$$f(x_1, ..., x_d|\theta) = c(u_1, ..., u_d|\theta_C) \prod_{i=1}^{n} f(x_i|\theta_i)$$

Bivariate copulas - Elliptical copulas

Gaussian Copula $C_R^{Ga}(u) = \Phi_R^n(\Phi^{-1}(u_1), .., \Phi^{-1}(u_d))$

Student Copula $C_R^{St}(u;\nu) = F_R^{MSt}(F^{-1}(u_1;\nu), .., F^{-1}(u_d;\nu))$

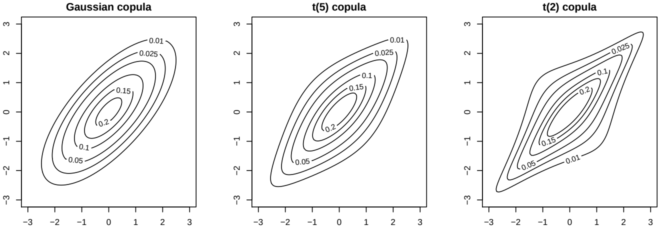


Figure: Contours of bivariate distributions with the same marginal standard normal

Bivariate copulas - Archimedean copulas

Common Bivariate Archimedean Copulas:

$$C(u_1, u_2) = \varphi^{-1}(\varphi(u_1) + \varphi(u_2))$$

Clayton (1978)

$\theta \geqslant 0$

$\varphi(t) = t^{-\theta} - 1$

$\varphi^{-1}(s) = (1+s)^{-1/\theta}$

Lower tail $\lambda_L = 2^{-1/\theta}$

Upper tail $\lambda_U = 0$

Frank (1979)

$\theta \geqslant 0$

$\varphi(t) = -ln \frac{e^{-\theta t} - 1}{e^{-\theta} - 1}$

$\varphi^{-1}(s) = -\frac{ln(1+e^{-s}(e^{-\theta}-1))}{\theta}$

No tail dependence

Gumbel (1960)

$\theta \geqslant 1$

$\varphi(t) = (-lnt)^{\theta}$

$\varphi^{-1}(s) = exp(-s^{1/\theta})$

Lower tail $\lambda_L = 0$

Upper tail $\lambda_U = 2 - 2^{1/\theta}$

Joe (1993)

$\theta \geqslant 1$

$\varphi(t) = -log(1-(1-t)^{\theta})$

$\varphi^{-1}(s) = (1-(1-e^{-s}))^{1/\theta}$

Lower tail $\lambda_L = 0$

Upper tail $\lambda_U = 2 - 2^{1/\theta}$

$$c_{sym}(u_1, u_2) = 0.5c(u_1, u_2) + 0.5c_{R180}(u_1, u_2)$$

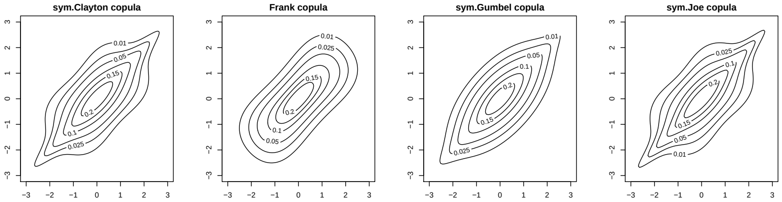


Figure: Contours of bivariate distributions with the same marginal standard normal

A copula model

Let (r_{1t}, r_{2t}), for $t = 1, \ldots, T$, be the time series of stock returns and bond returns that we want to model their joint dependence. Also let $F_1(r_{1t})$ and $F_2(r_{2t})$ be their marginal cumulative distribution function. Sklar (1959) decomposes the joint density function $f(r_{1t}, r_{2t})$ in terms of their marginal density functions (f_1, f_2) and copula density function c_{12} as

$$f(r_{1t}, r_{2t}) = f_1(r_{1t}) f_2(r_{2t}) c_{12}(u_{1t}, u_{2t}), \tag{1}$$

where $u_{1t} = F_1(r_{1t})$, and $u_{2t} = F_2(r_{2t})$, for $t = 1, \ldots, T$, are a sequence of independent random variables with uniform marginal distribution.

A generalized autoregressive score copula model

Creal et al. (2013) and Harvey (2013) propose dynamic models where time varying parameter follow a generalized autoregressive score (GAS),

$$\begin{aligned}(u_{1t}, u_{2t}) &\sim c_t(u_{1t}, u_{2t}|\theta_t),\\ \theta_t &= \Lambda(\lambda_t),\\ \lambda_{t+1} &= \lambda_0(1-\beta) + \alpha \frac{\partial \log c_t(u_{1t}, u_{2t}|\lambda_t)}{\partial \lambda_t} + \beta\lambda_t.\end{aligned} \tag{2}$$

Note that $0 < \beta < 1$. In the GAS copula model, the scores depends on the complete density rather than on its first or second moment. Blasques et al. (2014) proved that the use of the scores leads to the minimum Kullback-Leibler divergence between the true conditional density and the model-implied conditional density, while Koopman et al. (2016) showed some empirical examples where the GAS model out performs other observation driven models.

A GAS-MIDAS copula model

The copula parameters are assumed to follow a generalized autoregressive score (GAS) process.

$$
\begin{aligned}
(u_{1t}, u_{2t}) &\sim c_t(u_{1t}, u_{2t}|\theta_t), \\
\theta_t &= \Lambda(\lambda_t), \\
\lambda_{t+1} &= \lambda_\tau(1-\beta) + \alpha \frac{\partial \log c_t(u_{1t}, u_{2t}|\lambda_t)}{\partial \lambda_t} + \beta\lambda_t, \\
\lambda_\tau &= \lambda_0 + \sum_{j=1}^{N} \delta_j \left[\sum_{k=1}^{K_j} \phi_k(\omega_j) X_{j,\tau-k} \right],
\end{aligned} \tag{3}
$$

where $\tau = \lfloor t/L \rfloor$ and $(X_{1\tau}, \ldots, X_{N\tau})$ are N-dimensional vector of low frequency variables, and $\phi_k(\omega_j)$ is the weighting scheme of the variable j on its k lag, for $k = 1, \ldots, K$. The weighting scheme of each variable j depends on the regulated parameter ω_j for $j = 1, \ldots, N$.

An asymmetric GAS-MIDAS copula model

An asymmetric GAS-MIDAS copula model

$$
\begin{aligned}
(u_{1t}, u_{2t}) &\sim c_t(u_{1t}, u_{2t}|\theta_t), \\
\theta_t &= \Lambda(\lambda_t), \\
\lambda_{t+1} &= \lambda_\tau(1-\beta) + \alpha \frac{\partial \log c_t(u_{1t}, u_{2t}|\lambda_t)}{\partial \lambda_t} + \beta\lambda_t + \gamma\left(v_t - \bar{v}\right),
\end{aligned} \tag{4}
$$

where γ is the parameter that controls for the asymmetry, v_t is a measure of association related to “bad news” at time t and $\bar{v} = \mathbf{E}(v_t)$ at different quantiles $0 \leqslant q_1, q_2 \leqslant 1$ such as,

(a) Normal score: $v_t = \left[\Phi^{-1}(u_{1t})\mathbf{I}_{\{u_{1t}<q_1\}}\right]\left[\Phi^{-1}(u_{2t})\mathbf{I}_{\{u_{2t}<q_2\}}\right]$.

(b) Spearman's rank: $v_t = \left[(u_{1t}-0.5)\mathbf{I}_{\{u_{1t}<q_1\}}\right]\left[(u_{2t}-0.5)\mathbf{I}_{\{u_{2t}<q_2\}}\right]$.

(c) Spearman's footrule: $v_t = |u_{1t} - u_{2t}|\,\mathbf{I}_{\{u_{1t}<q_1\}}\mathbf{I}_{\{u_{2t}<q_2\}}$.

(d) Gini's gamma: $v_t = (|1-u_{1t}-u_{2t}| - |u_{1t}-u_{2t}|)\mathbf{I}_{\{u_{1t}<q_1\}}\mathbf{I}_{\{u_{2t}<q_2\}}$

Empirical illustration

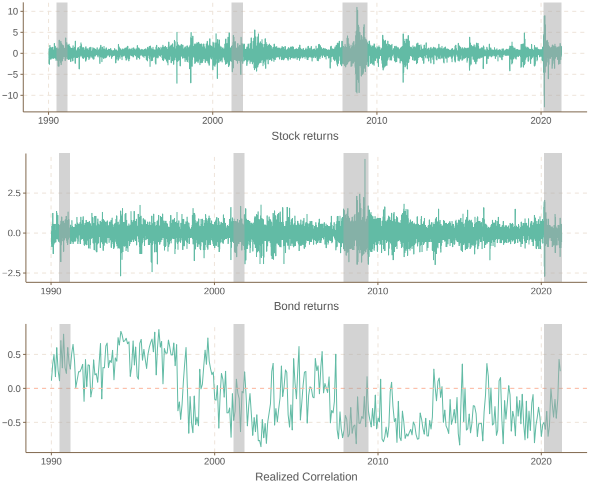


Figure: Stock returns and 10Y Government Bond returns.

Multiplicative GARCH MIDAS model, Conrad and Kleen (2020)

Let r_{it} be a return of Stock (or Bond) at time t,

$$r_{it} = \mu_i + \sqrt{\kappa_{i\tau} g_{it}} \epsilon_{it}$$

Multiplicative GARCH MIDAS models

$$g_{it} = (1 - \alpha_i - 0.5\gamma_i - \beta_i) + (\alpha_i + \gamma_i 1_{\{\epsilon_{i,t-1}<0\}}) g_{i,t-1} \epsilon^2_{i,t-1} + \beta_i g_{i,t-1}$$

GJR-GARCH

$$\kappa_{i\tau} = \exp\left(m_i + \sum_{j=1}^{N_i} \delta_{i,j} \left[\sum_{k=1}^{K_j} \phi_k(\omega_{i,j,1}, \omega_{i,j,2}) X_{i,j,\tau-k} \right] \right)$$

Long term components

(a) GARCH-MIDAS for the marginal distribution of Stock returns (01/01/1990 - 31/03/2021)

	μ	α	β	γ	m	δ_1	w_1	δ_2	w_2	K	LLH	BIC
RV Stock	0.029***	0.000	0.828***	0.215***	-1.243***	1.171***	2.840***			264	-1.301	2.610
	(0.009)	(0.009)	(0.016)	(0.023)	(0.119)	(0.091)	(0.665)					
VXO	0.026***	0.000	0.854***	0.086***	-2.057***	1.435***	3.789***			3	-1.286	2.580
	(0.009)	(0.014)	(0.022)	(0.018)	(0.075)	(0.052)	(0.743)					
VIX	0.025***	0.000	0.849***	0.098***	-2.158***	1.547***	3.417***			3	-1.289	2.586
	(0.009)	(0.012)	(0.023)	(0.019)	(0.085)	(0.061)	(0.627)					
VXO+NFCI	0.025***	0.000	0.851***	0.091***	-1.898***	1.355***	3.768***	0.151**	2.176	52	-1.286	2.582
	(0.009)	(0.014)	(0.023)	(0.018)	(0.102)	(0.063)	(0.723)	(0.064)	(1.659)			
VXO+NAI	0.024***	0.000	0.858***	0.091***	-1.936***	1.323***	3.766***	-0.197	7.039	36	-1.286	2.582
	(0.009)	(0.013)	(0.022)	(0.017)	(0.088)	(0.065)	(0.71)	(0.098)	(5.07)			
VXO+INDPRO	0.026***	0.000	0.847***	0.087***	-2.085***	1.463***	3.787***	-0.008	33.645	36	-1.286	2.582
	(0.009)	(0.014)	(0.024)	(0.018)	(0.089)	(0.057)	(0.753)	(0.009)	(45.377)			

The tables report the estimation results of the multiplicative component GARCH-MIDAS model for the Stock returns and Bond returns proposed by Conrad and Kleen (2020). The lag length K of the explanatory variables are set based on Conrad and Kleen (2020) and the weighting scheme is the restricted beta function. The values of the maximum likelihood (LLH) and the Bayesian information criteria (BIC) are normalized for the number of observations which shows that the GARCH-MIDAS with VXO index is preferred for the marginal distribution.

Empirical illustration - Copula functions

Table: Comparison of DCC MIDAS RC and GAS MIDAS RC Copulas

	α	β	λ_0	δ_1	$\omega_1^{(2)}$	ν	K	LLH	AIC	BIC
DCC	0.036***	0.960***						674.2	-1344.5	-1330.5
	(0.004)	(0.005)								
DCC MIDAS	0.065***	0.862***	0.013	1.009***	6.686***		24	702.4	-1394.8	-1359.9
	(0.007)	(0.023)	(0.019)	(0.050)	(1.806)					
GAS MIDAS Gaussian	0.213***	0.927***	0.011	1.987***	6.392***		24	701.0	-1392.0	-1357.1
	(0.023)	(0.018)	(0.039)	(0.105)	(1.705)					
GAS MIDAS Student	0.253***	0.934***	0.006	2.016***	6.410***	8.649***	24	759.8	-1507.6	-1465.8
	(0.032)	(0.020)	(0.045)	(0.122)	(1.903)	(0.937)				
GAS MIDAS sClayton	0.183***	0.961***	-0.032***	1.489***	2.933***		24	717.8	-1425.6	-1390.7
	(0.004)	(0.001)	(0.002)	(0.021)	(0.045)					
GAS MIDAS sGumbel	0.045***	0.958***	0.014***	0.881***	3.122***		24	735.5	-1461.0	-1426.2
	(0.000)	(0.000)	(0.001)	(0.005)	(0.017)					
GAS MIDAS Frank	1.624***	0.986***	-0.257***	4.852***	1.006***		24	653.2	-1296.3	-1261.5
	(0.024)	(0.000)	(0.004)	(0.027)	(0.000)					
GAS MIDAS sJoe	0.155***	0.969***	-0.035***	1.037***	2.094***		24	699.0	-1387.9	-1353.1
	(0.001)	(0.000)	(0.001)	(0.007)	(0.008)					

The table reports the estimation results of the DCC MIDAS and the GAS MIDAS copula model for the dependence of Stock returns and Bond returns in comparison to the benchmark DCC model. We choose the Realized correlation with the restricted beta weighting scheme function to explain for the long term component of the dependence. The lag length are selected such that the maximum likelihood becomes insensitive to the choice of K. The values of the LLH, the AIC, the BIC show that the GAS-MIDAS Student-t copula model is preferred for the dynamic dependence of Stock returns and Bond returns. ***,**,* denote significant at 1%, 5%, 10% level.

Factors that affect the Stock Bond dependence (I)

Table: Correlation matrix of explanatory variables

	RCor	PC II	PC SE	PC UC	PC IL	PC SPF II	PC SPF SE	PC SPF UC
PC II	0.429					0.807	0.046	-0.119
PC SE	0.264	0.375				0.296	0.432	-0.636
PC UC	-0.277	-0.175	-0.492			-0.086	-0.312	0.355
PC IL	-0.437	-0.083	-0.368	0.435		-0.087	-0.106	0.231
Inflation	0.323	0.789				0.725	0.084	-0.014
Term spread	0.013	-0.607				-0.277	0.025	0.086
Short-term interest	0.595	0.900				0.793	0.034	-0.173
Industrial Production	0.246		0.935			0.319	0.465	-0.586
ADS Index	0.114		0.363			-0.009	0.131	-0.041
Coincident Index	0.228		0.892			0.270	0.332	-0.657
VXO	-0.253			0.936		0.001	-0.320	0.380
RV Stock	-0.308			0.947		-0.032	-0.292	0.333
RV Bond	-0.176			0.805		-0.222	-0.221	0.232
Stock Illiquidity	-0.267				0.640	-0.041	-0.271	0.309
Bond Illiquidity	0.307				-0.672	0.072	-0.125	-0.001

The table reports the correlation matrix of explanatory variables. We divide 12 varibles into 4 main groups such as Inflation and Interest rate, the State of Economy, Uncertainty and Illiquidity.

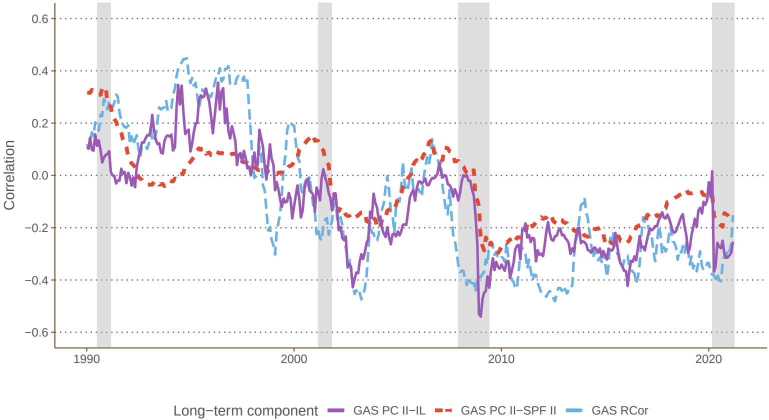
Factors that affect the Stock Bond dependence (II)

Table: The asymmetric GAS MIDAS Student-t Copula with one explanatory variable

	α	β	λ_0	γ	δ_1	$\omega_1^{(2)}$	ν	K	LLH	AIC	BIC
RCor	0.118***	0.904***	-0.056*	1.576***	1.514***	5.285***	9.2***	24	788.5	-1563.1	-1514.3
	(0.028)	(0.020)	(0.033)	(0.243)	(0.119)	(1.623)	(1.071)				
PC II	0.105***	0.980***	-0.209***	0.735***	0.152***	2.815	8.5***	12	763.0	-1511.9	-1463.1
	(0.019)	(0.004)	(0.051)	(0.136)	(0.044)	(3.250)	(0.917)				
PC SE	0.108***	0.982***	-0.220***	0.697***	0.088*	2.605***	8.5***	12	759.2	-1504.4	-1455.7
	(0.019)	(0.003)	(0.055)	(0.131)	(0.048)	(0.008)	(0.904)				
PC UC	0.109***	0.984***	-0.226***	0.652***	-0.021	7.592***	8.4***	18	757.7	-1501.4	-1452.6
	(0.019)	(0.003)	(0.059)	(0.126)	(0.053)	(0.046)	(0.890)				
PC IL	0.111***	0.976***	-0.214***	0.802***	-0.358***	5.075***	8.9***	18	764.2	-1514.4	-1465.6
	(0.019)	(0.005)	(0.048)	(0.158)	(0.085)	(0.079)	(1.001)				
PC SPF II	0.125***	0.970***	-0.126**	0.867***	0.282***	2.960	8.3***	6	765.7	-1517.5	-1468.7
	(0.022)	(0.007)	(0.050)	(0.167)	(0.075)	(2.723)	(0.876)				
PC SPF SE	0.111***	0.984***	-0.231***	0.645***	0.037	6.541***	8.3***	5	757.6	-1501.2	-1452.4
	(0.018)	(0.003)	(0.061)	(0.132)	(0.067)	(0.023)	(0.871)				
PC SPF UC	0.110***	0.984***	-0.231***	0.649***	-0.102	5.422	8.5***	4	758.1	-1502.2	-1453.5
	(0.018)	(0.003)	(0.059)	(0.125)	(0.121)	(6.115)	(0.902)				

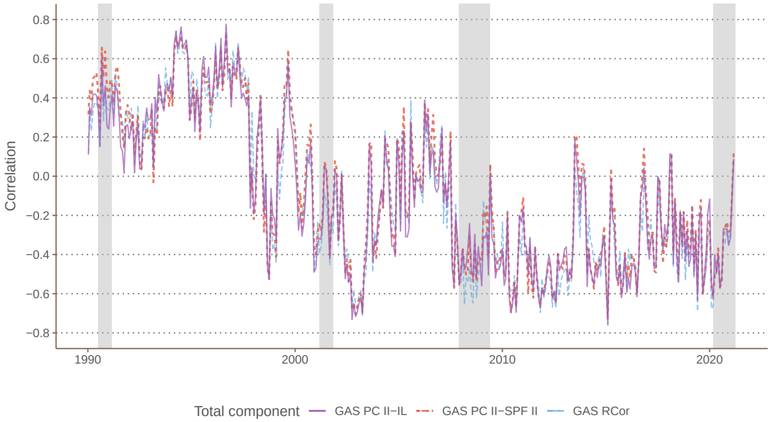
The table reports the estimation results of the asymmetric GAS MIDAS Student-t copula model for the dependence of Stock returns and Bond returns. We choose one explanatory variable with the restricted beta weighting scheme function to explain for the long term component of the dependence. The lag length are selected such that the maximum likelihood becomes insensitive to the choice of K. The values of the LLH, the AIC, the BIC show that Realized correlation is the most preferred for the dynamic dependence of Stock returns and Bond returns. ***,**,* denote significant at 1%, 5%, 10% level.

Long-term dependence component



The figure shows the long-term Kendall's τ correlation between Stock returns and Bond returns using DCC MIDAS with RC, GAS MIDAS with RC, GAS MIDAS with SPF, GAS MIDAS with RC and SPF. The shaded areas highlight the recession periods based on the NBER indicators.

Total dependence



The figure shows the Kendall's τ correlation between Stock returns and Bond returns using DCC MIDAS with RC, GAS MIDAS with RC, GAS MIDAS with SPF, GAS MIDAS with RC and SPF. The shaded areas highlight the recession periods based on the NBER indicators.

Out-of-sample forecast - VaR

Based on the simulated returns, we construct the simulated portfolio of stock and bond at time t and calculate the $\mathrm{VaR}_{q,t}$ and $\mathrm{ES}_{q,t}$ and their associate risk measure,

$$
\begin{aligned}
q &= Pr\left(r_t \leq \mathrm{VaR}_{q,t}\right), \\
\mathrm{ES}_{q,t} &= E\left(r_t \,|r_t \leq \mathrm{VaR}_{q,t}\,\right), \\
\mathsf{IF} &= \sum_{t=1}^{T} \mathbf{I}\left(r_t < \mathrm{VaR}_{q,t}\right), \\
\mathsf{AD}_t &= ||r_t| - |\,\mathrm{VaR}_{q,t}\,||\; \mathbf{I}\left(r_t < \mathrm{VaR}_{q,t}\right), \\
\mathsf{SD}_t &= (|r_t| - |\,\mathrm{VaR}_{q,t}\,|)^2\; \mathbf{I}\left(r_t < \mathrm{VaR}_{q,t}\right), \\
\mathsf{QS}_t &= (r_t - \mathrm{VaR}_{q,t})(q - \mathbf{I}\left(r_t < \mathrm{VaR}_{q,t}\right)), \\
\mathsf{ALS}_t &= -\log\left(\frac{q}{\mathrm{ES}_{q,t}}\right) - \frac{(r_t - \mathrm{VaR}_{q,t})(q - \mathbf{I}\left(r_t < \mathrm{VaR}_{q,t}\right))}{q\,\mathrm{ES}_{q,t}}.
\end{aligned}
$$

Out-of-sample forecast - VaR

Table: Risk measures

	VaR	ES	IF	AD	SD	QS	ALS
				q = 1%			
DCC	-1.291	-1.644	14	6.149	4.212	19.383	1527.997
DCC MIDAS	-1.286	-1.641	14	6.071	4.123*	19.256*	1519.746
GAS MIDAS RCor	-1.302	-1.673	14	5.671	3.752	19.020	1503.488
GAS MIDAS PC II	-1.321	-1.689	12	5.461**	3.489**	18.991*	1495.735
GAS MIDAS PC II-IL	-1.325	-1.696	12	5.187**	3.241**	18.762*	1470.668
				q = 0.5 %			
DCC	-1.497	-1.905	10	4.023	2.366	11.670	1713.927
DCC MIDAS	-1.492	-1.905	10	3.745	2.092	11.365	1684.572
GAS MIDAS RCor	-1.519	-1.945	10	3.502	1.937	11.259	1669.379
GAS MIDAS PC II	-1.541	-1.959	8	3.043**	1.543*	10.910	1629.636
GAS MIDAS PC II-IL	-1.543	-1.970	7	2.858**	1.504**	10.733	1600.582

The table reports the average VaR and ES together with the sum of the associated risk measures. ***,**,* denote that the corresponding model significantly outperforms the Gaussian VAR at 1%, 5%, 10% level.

Out-of-sample forecast - Portfolio allocation

Following Patton (2004), we assume the CRRA utility function as,

$$U(r_t,\eta) = \begin{cases} (1-\eta)^{-1}\left(P_0(1+r_t)\right)^{1-\eta} & \text{if } \eta \neq 1 \\ \log\left(P_0(1+r_t)\right) & \text{if } \eta = 1 \end{cases} \tag{5}$$

$$r_t = w_{1t}r_{1t} + (1-w_{1t})r_{2t}$$

We measure the performance fee (Fee) and the break even transaction cost (TC) per trade,

$$\sum_{t=1}^{T} U(r_t^B - \text{Fee}, \eta) = \sum_{t=1}^{T} U(r_t^A, \eta),$$

$$\sum_{t=1}^{T} U\left(r_t^B - TC\left|w_{1t}^B - w_{1,t-1}^B \frac{1+r_{1t-1}}{1+r_{t-1}^B}\right|, \eta\right) = \sum_{t=1}^{T} U\left(r_t^A, \eta\right)$$

Out-of-sample forecast - Portfolio allocation

Table: Economic values of dynamic portfolios over a passive portfolio

	$\eta = 1$		$\eta = 5$		$\eta = 10$	
	Fee	TC	Fee	TC	Fee	TC
DCC	365.61	16.83	148.68	23.91	121.93	26.73
DCC MIDAS	381.18	17.54	157.37	25.54	134.12	30.13
GAS MIDAS RCor	393.09	17.62	161.45	26.15	137.43	30.91
GAS MIDAS PC II	365.88	16.25	154.10	24.48	133.08	29.64
GAS MIDAS PC II-IL	370.62	15.99	161.97	25.57	140.58	31.31

The table reports the economic values of dynamic portfolios over a passive portfolio. The initial weight of the passive portfolio is chosen to maximize the CRRA utility function using the historical in-sample data. The performance fees are normalized to annual basis points (bps) and the break even transaction costs are expressed in basis points of the proportional cost for reweighting.

Contributions and Discussions

Contributions

A Generalized Autoregressive Score (GAS) Mixed Data Sampling (MIDAS) copula model is proposed for the dynamic dependence of Stock returns and Bond returns.

Findings

Besides the realized covariance, other economic variables can explain for the change in the long term dependence of Stock returns and Bond returns such as Inflation and Interest rate, the state of economy, etc...

Moreover, the Survey of Professional forecasters can provide a forward looking for the changes in the dependence.

Thank you

A Dynamic Conditional Correlation Gaussian copula model

Following Engle (2002), a DCC model can be presented as,

$$\begin{aligned}(\epsilon_{1t}, \epsilon_{2t}) &\sim \mathbf{N}(\epsilon_{1t}, \epsilon_{2t}|\mathbf{0}, R_t), \\ R_t &= Q_t^{*-1/2} Q_t Q_t^{*-1/2} \text{ where } Q_t^* = diag(Q_t) \\ q_{12,t+1} &= q_{12,0}(1-\alpha-\beta) + \alpha\epsilon_{1t}\epsilon_{2t} + \beta q_{12,t},\end{aligned} \tag{6}$$

where $(q_{12,0}, \alpha, \beta)$ are the fixed parameters (Note that $0 < \alpha + \beta < 1$). A equivalent DCC Gaussian copula,

$$(u_{1t}, u_{2t}) \sim c_{12}^{(Gauss)}(u_{1t}, u_{2t}|R_t), \tag{7}$$

where $\epsilon_{1t} = \Phi^{-1}(u_{1t}), \epsilon_{2t} = \Phi^{-1}(u_{2t})$.

A DCC MIDAS Gaussian copula model

Colacito et al. (2011) and Conrad et al. (2014) extend the DCC model such that there are N variables that can explain for the long term dependence. A DCC MIDAS model can be presented as,

$$
\begin{aligned}
(\epsilon_{1t}, \epsilon_{2t}) &\sim \mathbf{N}(\epsilon_{1t}, \epsilon_{2t} | \mathbf{0}, R_t), \\
R_t &= Q_t^{*-1/2} Q_t Q_t^{*-1/2} \text{ where } Q_t^* = diag(Q_t), \\
q_{12,t+1} &= q_{12,\tau}(1-\alpha-\beta) + \alpha\epsilon_{1t}\epsilon_{2t} + \beta q_{12,t}, \\
q_{12,\tau} &= \lambda_0 + \sum_{j=1}^{N} \delta_j \left[\sum_{k=1}^{K_j} \phi_k(\omega_{j,1}, \omega_{j,2}) X_{j,\tau-k} \right],
\end{aligned} \tag{8}
$$

where $(\lambda_0, \alpha, \beta, \delta_j, \omega_j)$ are the fixed copula parameters and $\tau = \lfloor t/L \rfloor$. $(X_{1\tau}, \dots, X_{N\tau})$ are N-dimensional vector of low frequency variables, and $\phi_k(\omega_{j1}, \omega_{j2})$ is the weighting scheme of the variable j on its k lag, for $k = 1, \dots, K$. Note that $0 < \alpha + \beta < 1$.

Simulation studies - Dynamic update

We compare the proposed GAS MIDAS RC copula models with the EWMA, the DCC (Engle, 2002), the Gaussian GAS (Creal et al., 2013) model in different stress scenarios based on the proposal of Engle (2002). We simulate $T = 2000$ observations from a bivariate Gaussian copula with time-varying correlation parameter ρ_t. We consider five models for the behavior of ρ_t such that,

(a) Constant: $\rho_t = 0.8$.

(b) Sine: $\rho_t = 0.5cos(2\pi t/250)$.

(c) Fast Sine: $\rho_t = 0.5cos(2\pi t/25)$.

(d) Step: $\rho_t = 0.5 - I(t > 1000)$.

(e) Ramp: $\rho_t = ((t \mod 200) - 100)/101$.

Simulation studies - Number of lags

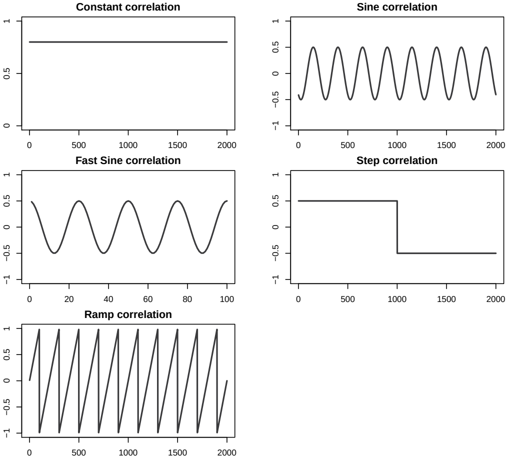


Figure: The ρ_t processes for different test scenarios

Simulation studies - MSE and MAE

Table: MAE and MSE results: a simulation study

	Constant	Sine	Fast sine	Step	Ramp
			(a) MAE		
EWMA	5.067	1.184	1.180	1.003	1.357
DCC	1.000	1.000	1.000	1.000	1.000
GAS	0.798	1.002	0.988	0.875	0.940
GAS MIDAS	0.943	0.609	0.986	0.866	0.875
			(b) MSE		
EWMA	25.007	1.341	1.399	0.868	1.902
DCC	1.000	1.000	1.000	1.000	1.000
GAS	0.666	0.999	0.978	0.863	0.919
GAS MIDAS	0.980	0.387	0.978	0.850	0.828

The table shows the relative MAE and MSE of the estimation of the correlation using the EWMA, the GAS, the GAS MIDAS over the benchmark DCC model. We use the restricted beta weighting function and $K = 9$ lags of monthly RCor as a low frequency explanatory variable for the long term change in the correlation. We generate 200 psuedo datasets for each stress test and calculate the average of MAE and MSE. The entries less than 1 indicate that the given model is better.

Factors that affect the Stock Bond dependence (III)

Table: The GAS MIDAS Student-t Copula with two explanatory variables

	α	β	λ_0	γ	δ_1	$\omega_1^{(2)}$	δ_2	$\omega_2^{(2)}$	ν	K	LLH	AIC	BIC
(a) Realized Cor. with													
PC II	0.124***	0.908***	-0.068**	1.519***	1.476***	4.575**	0.022	1.330	9.2***	12	788.7	-1559.4	-1496.6
	(0.027)	(0.021)	(0.034)	(0.242)	(0.138)	(1.869)	(0.035)	(4.174)	(1.070)				
PC SE	0.126***	0.904***	-0.064*	1.516***	1.497***	5.725***	0.025	1.740	8.9***	12	788.9	-1559.7	-1497.0
	(0.028)	(0.020)	(0.034)	(0.238)	(0.127)	(1.783)	(0.033)	(4.084)	(1.001)				
PC UC	0.121***	0.906***	-0.066*	1.594***	1.496***	4.437**	-0.010	1.345	8.9***	18	788.3	-1558.6	-1495.9
	(0.027)	(0.021)	(0.035)	(0.251)	(0.140)	(1.755)	(0.069)	(3.630)	(0.997)				
PC IL	0.112***	0.904***	-0.103***	1.594***	1.270***	6.756***	-0.198**	1.628*	9.4***	18	791.4	-1564.7	-1502.0
	(0.027)	(0.020)	(0.035)	(0.250)	(0.146)	(2.320)	(0.080)	(0.912)	(1.113)				
PC SPF II	0.108***	0.931***	-0.056*	1.342***	1.495***	2.139***	-0.017	3.909***	9.3***	6	785.1	-1552.2	-1489.5
	(0.025)	(0.012)	(0.033)	(0.210)	(0.177)	(0.095)	(0.049)	(0.022)	(1.097)				
PC SPF SE	0.116***	0.928***	-0.093**	1.264***	1.463***	3.302	-0.006	2.921***	8.4***	5	786.5	-1555.0	-1492.3
	(0.030)	(0.023)	(0.043)	(0.221)	(0.172)	(2.136)	(0.043)	(0.267)	(0.884)				
PC SPF UC	0.118***	0.902***	-0.067**	1.590***	1.525***	5.386***	-0.042	2.454***	9.1***	4	788.6	-1559.3	-1496.5
	(0.028)	(0.020)	(0.033)	(0.247)	(0.118)	(1.653)	(0.072)	(0.025)	(1.050)				

The table reports the estimation results of the asymmetric GAS MIDAS Student-t copula model for the dependence of Stock returns and Bond returns. We choose the Realized correlation with another explanatory variable to explain for the long term component of the dependence. The lag length are selected such that the maximum likelihood becomes insensitive to the choice of K and the restricted beta weighting scheme function is chosen based on previous analysis. The values of the LLH, the AIC, the BIC show that Realized correlation is the most preferred for the dynamic dependence of Stock returns and Bond returns. ***,**,* denote significant at 1%, 5%, 10% level.

Factors that affect the Stock Bond dependence (IV)

Table: The GAS MIDAS Student-t Copula with two explanatory variables

	α	β	λ_0	γ	δ_1	$\omega_1^{(2)}$	δ_2	$\omega_2^{(2)}$	ν	K	LLH	AIC	BIC
(b) PC Inflation with													
PC SE	0.105***	0.978***	-0.206***	0.772***	0.144***	6.347***	0.045	2.556***	8.4***	12	763.3	-1508.6	-1445.9
	(0.019)	(0.004)	(0.049)	(0.139)	(0.045)	(0.021)	(0.045)	(0.008)	(0.897)				
PC UC	0.105***	0.979***	-0.208***	0.741***	0.145***	6.373***	-0.016	2.549***	8.5***	18	763.0	-1508.0	-1445.3
	(0.019)	(0.004)	(0.051)	(0.137)	(0.044)	(0.039)	(0.051)	(0.009)	(0.921)				
PC IL	0.104***	0.956***	-0.197***	1.181***	0.197***	1.326	-0.405***	7.399	9.1***	18	778.0	-1538.1	-1475.4
	(0.021)	(0.009)	(0.036)	(0.199)	(0.031)	(0.986)	(0.087)	(6.484)	(1.057)				
PC SPF II	0.130***	0.977***	-0.055	0.717***	0.134***	1.813	0.134***	1.813	9.6***	6	760.8	-1507.7	-1458.0
	(0.021)	(0.005)	(0.067)	(0.178)	(0.028)	(1.124)	(0.028)	(1.124)	(1.182)				
PC SPF SE	0.137***	0.983***	0.009	0.549**	0.198***	3.025	0.109	2.601***	8.6***	5	756.1	-1494.2	-1431.5
	(0.025)	(0.007)	(0.145)	(0.242)	(0.059)	(3.481)	(0.104)	(0.051)	(0.950)				
PC SPF UC	0.107***	0.977***	-0.192***	0.874***	0.137***	8.411***	-0.104	2.445***	8.2***	4	762.4	-1506.8	-1444.1
	(0.018)	(0.005)	(0.048)	(0.162)	(0.041)	(0.031)	(0.123)	(0.016)	(0.840)				

The table reports the estimation results of the asymmetric GAS MIDAS Student-t copula model for the dependence of Stock returns and Bond returns. We choose the Realized correlation with another explanatory variable to explain for the long term component of the dependence. The lag length are selected such that the maximum likelihood becomes insensitive to the choice of K and the restricted beta weighting scheme function is chosen based on previous analysis. The values of the LLH, the AIC, the BIC show that Realized correlation is the most preferred for the dynamic dependence of Stock returns and Bond returns. ***,**,* denote significant at 1%, 5%, 10% level.

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